



# Modified Noor iterations with errors for nonlinear equations in Banach spaces

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## Abstract

We introduce a new three step iterative scheme with errors to approximate the unique common fixed point of a family of three strongly pseudocontractive (accretive) mappings on Banach spaces. Our results are generalizations and improvements of results obtained by several authors in literature. In particular, they generalize and improve the results of Mogbademu and Olaleru [A. A. Mogbademu and J. O. Olaleru, Bull. Math. Anal. Appl., **3** (2011), 132-139], Xue and Fan [Z. Xue and R. Fan, Appl. Math. Comput., **206** (2008), 12-15] which is in turn a correction of Rafiq [A. Rafiq, Appl. Math. Comput., **182** (2006), 589-595]. ©2014 All rights reserved.

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## 1. Introduction and Preliminaries

Let  $E$  be a real Banach space with dual  $E^*$  and  $D$  is a nonempty closed convex subset of  $E$ . We denote by  $J$  the normalized duality from  $E$  to  $2^{E^*}$  defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing. We shall also denote the single-valued duality mapping by  $j$ .

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**Definition 1.1** [20]. A map  $T : E \rightarrow E$  is called strongly accretive if there exists a constant  $k > 0$  such that, for each  $x, y \in E$ , there is a  $j(x - y) \in J(x - y)$  satisfying

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2. \quad (1.2)$$

**Definition 1.2** [20]. An operator  $T$  with domain  $D(T)$  and range  $R(T)$  in  $E$  is called *strongly pseudocontractive* if for all  $x, y \in D(T)$ , there exists  $j(x - y) \in J(x - y)$  and a constant  $0 < k < 1$  such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2. \quad (1.3)$$

The class of strongly accretive operators is closely related to the class of strongly pseudocontractive operators. It is well known that  $T$  is strongly pseudocontractive if and only if  $(I - T)$  is strongly accretive, where  $I$  denotes the identity operator. Browder [1] and Kato [8] independently introduced the concept of accretive operators in 1967. One of the early results in the theory of accretive operators credited to Browder states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0 \quad (1.4)$$

is solvable if  $T$  is locally Lipschitzian and accretive in an appropriate Banach space.

These class of operators have been studied extensively by several authors (see [2], [3], [9], [10], [11], [15], [16], [18], [20], [25]).

**Definition 1.3** [20]. A mapping  $T : E \rightarrow E$  is called Lipschitzian if there exists a constant  $L > 0$  such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in D(T). \quad (1.5)$$

In 1953, Mann [10] introduced the Mann iterative scheme and used it to prove the convergence of the sequence to the fixed points for which the Banach principle is not applicable. Later in 1974, Ishikawa [6] introduced an iterative process to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann iterative scheme failed to converge. In 2000 Noor [14] gave the following three-step iterative scheme (or Noor iteration) for solving nonlinear operator equations in uniformly smooth Banach spaces.

Let  $D$  be a nonempty convex subset of  $E$  and let  $T : D \rightarrow D$  be a mapping. For a given  $x_0 \in K$ , compute the sequence  $\{x_n\}_{n=0}^{\infty}$  by the iterative schemes

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ y_n = (1 - \beta_n)x_n + \beta_nTz_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \quad n \geq 0 \end{cases} \quad (1.6)$$

where  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$  and  $\{\gamma_n\}_{n=0}^{\infty}$  are three real sequences in  $[0, 1]$  satisfying some conditions.

If  $\gamma_n = 0$  and  $\beta_n = 0$ , for each  $n \in \mathbb{Z}$ ,  $n \geq 0$ , then (1.6) reduces to:

the iterative scheme

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \in \mathbb{Z}, \quad n \geq 0, \quad (1.7)$$

which is called the one-step (or Mann iterative scheme), introduced by Mann [9].

For  $\gamma_n = 0$ , (1.6) reduces to:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0 \end{cases} \quad (1.8)$$

where  $\{\alpha_n\}_{n=0}^{\infty}$  and  $\{\beta_n\}_{n=0}^{\infty}$  are two real sequences in  $[0, 1]$  satisfying some conditions. Equation (1.8) is called the two-step (or Ishikawa iterative process) introduced by Ishikawa [6].

In 1989, Glowinski and Le-Tallec [4] used a three-step iterative process to solve elastoviscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge *et al.* [5] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [4] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions. Hence, we can conclude by observing that three-step iterative scheme play an important role in solving various problems in pure and applied sciences. Studies in nonlinear functional analysis reveals that several problems in sciences, engineering and management sciences can be converted and solved as a fixed point problem of the form  $x = Tx$ , where  $T$  is a mapping. Several authors in literature have obtained some interesting fixed points results (see, e.g. [1, 7, 8, 12, 13, 21, 19, 24, 26, 27]).

Rafiq [20] recently introduced the following modified three-step iterative scheme and used it to approximate the unique common fixed point of a family of strongly pseudocontractive operators.

Let  $T_1, T_2, T_3 : D \rightarrow D$  be three given mappings. For a given  $x_0 \in D$ , compute the sequence  $\{x_n\}_{n=0}^\infty$  by the iterative scheme

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T_2 z_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases} \quad (1.9)$$

where  $\{\alpha_n\}_{n=0}^\infty$ ,  $\{\beta_n\}_{n=0}^\infty$  and  $\{\gamma_n\}_{n=0}^\infty$  are three real sequences in  $[0, 1]$  satisfying some conditions. Observe that iterative schemes (1.6)-(1.8) are special cases of (1.9).

More recently, Suantai [22] introduced the following three-step iterative schemes.

Let  $E$  be a normed space,  $D$  be a nonempty convex subset of  $E$  and  $T : D \rightarrow D$  be a given mapping. Then for a given  $x_1 \in D$ , compute the sequence  $\{x_n\}_{n=1}^\infty$ ,  $\{y_n\}_{n=1}^\infty$  and  $\{z_n\}_{n=1}^\infty$  by the iterative scheme

$$\begin{cases} z_n = a_n T^n x_n + (1 - a_n)x_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n)x_n, \quad n \geq 1, \end{cases} \quad (1.10)$$

where  $\{a_n\}_{n=1}^\infty$ ,  $\{b_n\}_{n=1}^\infty$ ,  $\{c_n\}_{n=1}^\infty$ ,  $\{\alpha_n\}_{n=1}^\infty$ ,  $\{\beta_n\}_{n=1}^\infty$  are appropriate sequences in  $[0, 1]$ .

Motivated by the facts above, we now introduce the following modified three-step iterative scheme with errors which we shall use in this paper to approximate the unique common fixed point of a family of strongly pseudocontractive maps.

Let  $E$  be a real Banach space,  $D$  be a nonempty convex subset of  $E$  and  $T_1, T_2, T_3 : D \rightarrow D$  be a family of three maps. Then for a given  $x_0, u_0, v_0, w_0 \in D$ , compute the sequence  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  by the iterative scheme

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n - e_n)x_n + \alpha_n T_1 y_n + \beta_n T_1 z_n + e_n u_n & n \geq 0 \\ y_n = (1 - a_n - b_n - e'_n)x_n + a_n T_2 z_n + b_n T_2 x_n + e'_n v_n \\ z_n = (1 - c_n - e''_n)x_n + c_n T_3 x_n + e''_n w_n, \end{cases} \quad (1.11)$$

where  $\{a_n\}_{n=0}^\infty$ ,  $\{b_n\}_{n=0}^\infty$ ,  $\{c_n\}_{n=0}^\infty$ ,  $\{\alpha_n\}_{n=0}^\infty$ ,  $\{\beta_n\}_{n=0}^\infty$ ,  $\{e_n\}_{n=0}^\infty$ ,  $\{e'_n\}_{n=0}^\infty$ ,  $\{e''_n\}_{n=0}^\infty$  are real sequences in  $[0, 1]$  satisfying certain conditions and  $\{u_n\}_{n=0}^\infty$ ,  $\{v_n\}_{n=0}^\infty$ ,  $\{w_n\}_{n=0}^\infty$  are bounded sequences in  $D$ .

Observe that (1.6)-(1.10) and the modified three step iteration process with errors introduced by Mogbademu and Olaleru [11] are special cases of (1.11). In this paper, we shall use algorithm (1.11) to approximate the unique common fixed point of a family of three strongly pseudocontractive operators in Banach spaces. Our results are generalizations and improvements of the results of Mogbademu and Olaleru [11], Xue and Fan [25] which in turn is a correction of Rafiq [20].

Rafiq [20] proved the following theorem

**Theorem R** [20]. Let  $E$  be a real Banach space and  $D$  be a nonempty closed convex subset of  $E$ . Let  $T_1, T_2, T_3$  be strongly pseudocontractive self maps of  $D$  with  $T_1(D)$  bounded and  $T_1, T_3$  be uniformly continuous. Let  $\{x_n\}_{n=0}^\infty$  be the sequence defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T_2 z_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases}$$

where  $\{\alpha_n\}_{n=0}^\infty$ ,  $\{\beta_n\}_{n=0}^\infty$  and  $\{\gamma_n\}_{n=0}^\infty$  are three real sequences in  $[0, 1]$  satisfying the conditions:

$$\lim_{n \rightarrow \infty} \alpha_n = 0 = \lim_{n \rightarrow \infty} \beta_n \text{ and } \sum_{n=0}^\infty \alpha_n = \infty.$$

If  $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ , then the sequence  $\{x_n\}_{n=0}^\infty$  converges strongly to the common fixed point of  $T_1, T_2, T_3$ .

Xue and Fan [25] obtained the following convergence results which in turn is a correction of Theorem R.

**Theorem XF** [25]. Let  $E$  be a real Banach space and  $D$  be a nonempty closed convex subset of  $E$ . Let  $T_1, T_2$  and  $T_3$  be strongly pseudocontractive self maps of  $D$  with  $T_1(D)$  bounded and  $T_1, T_2$  and  $T_3$  uniformly continuous. Let  $\{x_n\}_{n=0}^\infty$  be defined by (1.9), where  $\{\alpha_n\}_{n=0}^\infty$ ,  $\{\beta_n\}_{n=0}^\infty$  and  $\{\gamma_n\}_{n=0}^\infty$  are three real sequences in  $[0, 1]$  which satisfy the conditions:  $\alpha_n, \beta_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\sum_{n=0}^\infty \alpha_n = \infty$ . If  $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ , then the sequence  $\{x_n\}_{n=0}^\infty$  converges strongly to the common fixed point of  $T_1, T_2$  and  $T_3$ .

In this study, we use our newly introduced iterative scheme (1.11) to prove some convergence results. Our results are generalizations and improvements of the results of Mogbademu and Olaleru [11], Xue and Fan [25] which in turn is a correction of Rafiq [20].

The following lemmas will be useful in this study.

**Lemma 1.1** [20]. Let  $E$  be a real Banach space and  $J : E \rightarrow 2^{E^*}$  be the normalized duality mapping. Then, for any  $x, y \in E$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y). \quad (1.12)$$

**Lemma 1.2** [23]. Let  $\{\rho\}_{n=0}^\infty$  be a nonnegative sequence which satisfies the following inequality:

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n, \quad n \geq 0, \quad (1.13)$$

where  $\lambda_n \in (0, 1)$ ,  $n = 0, 1, 2, \dots$ ,  $\sum_{n=0}^\infty \lambda_n = \infty$  and  $\sigma_n = o(\lambda_n)$ . Then  $\rho_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 2. Main Results

**Theorem 2.1** Let  $E$  be a real Banach space and  $D$  be a nonempty closed convex subset of  $E$ . Let  $T_1, T_2$  and  $T_3$  be strongly pseudocontractive self maps of  $D$  with  $T_1(D)$  bounded and  $T_1, T_2$  and  $T_3$  uniformly continuous. Let  $\{x_n\}_{n=0}^\infty$  be defined by (1.11), where  $\{a_n\}_{n=0}^\infty$ ,  $\{b_n\}_{n=0}^\infty$ ,  $\{c_n\}_{n=0}^\infty$ ,  $\{\alpha_n\}_{n=0}^\infty$ ,  $\{\beta_n\}_{n=0}^\infty$ ,  $\{e_n\}_{n=0}^\infty$ ,  $\{e'_n\}_{n=0}^\infty$ ,  $\{e''_n\}_{n=0}^\infty$  are real sequences in  $[0, 1]$  satisfying the conditions:  $a_n, b_n, c_n, e_n, e'_n, e''_n, \alpha_n, \beta_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\alpha_n + \beta_n + e_n < 1$ ,  $a_n + b_n + e'_n < 1$ ,  $c_n + e''_n < 1$ ,  $\sum_{n=0}^\infty \alpha_n = \infty$  and  $\{u_n\}$ ,  $\{v_n\}$ ,  $\{w_n\}$  are bounded sequences in  $D$ . If  $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ , then the sequence  $\{x_n\}_{n=0}^\infty$  converges strongly to the common fixed point of  $T_1, T_2$  and  $T_3$ .

**Proof.** Since  $T_1, T_2, T_3$  are strongly pseudocontractive, there exists a constant  $k = \max\{k_1, k_2, k_3\}$  so that

$$\langle T_i x - T_i y, j(x - y) \rangle \leq k\|x - y\|^2, \quad i = 1, 2, 3, \quad (2.1)$$

where  $k_1, k_2$  and  $k_3$  are constants for operators  $T_1, T_2$  and  $T_3$  respectively. Assume that  $p \in F(T_1) \cap F(T_2) \cap F(T_3)$ , using the fact that  $T_i$  is strongly pseudocontractive for each  $i = 1, 2, 3$  we obtain  $F(T_1) \cap F(T_2) \cap F(T_3) = p \neq \emptyset$ . Since  $T_1$  has a bounded range, we let

$$M_1 = \|x_0 - p\| + \sup_{n \geq 0} \|T_1 y_n - p\| + \sup_{n \geq 0} \|T_1 z_n - p\| + \|u_n - p\|. \quad (2.2)$$

We shall prove by induction that  $\|x_n - p\| \leq M_1$  holds for all  $n \in \mathbb{N}$ . We observe from (2.2) that  $\|x_0 - p\| \leq M_1$ . Assume that  $\|x_n - p\| \leq M_1$  holds for all  $n \in \mathbb{N}$ . We will prove that  $\|x_{n+1} - p\| \leq M_1$ . Using (1.11), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n - e_n)(x_n - p) + \alpha_n(T_1 y_n - p) + \\ &\quad \beta_n(T_1 z_n - p) + e_n(u_n - p)\| \\ &\leq (1 - \alpha_n - \beta_n - e_n)\|x_n - p\| + \alpha_n\|T_1 y_n - p\| + \\ &\quad \beta_n\|T_1 z_n - p\| + e_n\|u_n - p\| \\ &\leq (1 - \alpha_n - \beta_n - e_n)M_1 + \alpha_n M_1 + \beta_n M_1 + e_n M_1 \\ &= M_1. \end{aligned} \quad (2.3)$$

Using the uniform continuity of  $T_3$ , we obtain that  $\{T_3 x_n\}_{n=0}^\infty$  is bounded. We now set

$$M_2 = \max \left\{ M_1, \sup_{n \geq 0} \|T_3 x_n - p\|, \sup_{n \geq 0} \|w_n - p\| \right\}, \quad (2.4)$$

hence

$$\begin{aligned} \|z_n - p\| &= \|(1 - c_n - e_n'')(x_n - p) + c_n(T_3 x_n - p) + e_n''(w_n - p)\| \\ &\leq (1 - c_n - e_n'')\|x_n - p\| + c_n\|T_3 x_n - p\| + e_n''\|w_n - p\| \\ &\leq (1 - c_n - e_n'')M_1 + c_n M_2 + e_n'' M_2 \\ &\leq (1 - c_n - e_n'')M_2 + c_n M_2 + e_n'' M_2 \\ &= M_2. \end{aligned} \quad (2.5)$$

By the uniform continuity of  $T_2$ , we obtain  $\{T_2 z_n\}_{n=0}^\infty$  and  $\{T_2 x_n\}_{n=0}^\infty$  are bounded. Set

$$M = \sup_{n \geq 0} \|T_2 z_n - p\| + \sup_{n \geq 0} \|x_n - p\| + \sup_{n \geq 0} \|v_n - p\| + M_2. \quad (2.6)$$

Using Lemma 1.1 and (1.11), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n - e_n)(x_n - p) + \alpha_n(T_1 y_n - p) + \\ &\quad \beta_n(T_1 z_n - p) + e_n(u_n - p)\|^2 \\ &\leq (1 - \alpha_n - \beta_n - e_n)^2 \|x_n - p\|^2 \\ &\quad + 2\langle \alpha_n(T_1 y_n - p) + \beta_n(T_1 z_n - p) + e_n(u_n - p), j(x_{n+1} - p) \rangle \\ &= (1 - \alpha_n - \beta_n - e_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T_1 y_n - p, j(x_{n+1} - p) \rangle \\ &\quad + 2\beta_n \langle T_1 z_n - p, j(x_{n+1} - p) \rangle + 2e_n \langle u_n - p, j(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n - \beta_n - e_n)^2 \|x_n - p\|^2 \\ &\quad + 2\alpha_n \langle T_1 x_{n+1} - T_1 p, j(x_{n+1} - p) \rangle \\ &\quad + 2\alpha_n \langle T_1 y_n - T_1 x_{n+1}, j(x_{n+1} - p) \rangle \\ &\quad + 2\beta_n \langle T_1 x_{n+1} - T_1 p, j(x_{n+1} - p) \rangle \\ &\quad + 2\beta_n \langle T_1 z_n - T_1 x_{n+1}, j(x_{n+1} - p) \rangle + 2e_n \langle u_n - p, j(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n - \beta_n - e_n)^2 \|x_n - p\|^2 + 2\alpha_n k \|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n \|T_1 y_n - T_1 x_{n+1}\| \cdot \|x_{n+1} - p\| + 2\beta_n k \|x_{n+1} - p\|^2 \\ &\quad + 2\beta_n \|T_1 z_n - T_1 x_{n+1}\| \cdot \|x_{n+1} - p\| + 2e_n M \\ &\leq (1 - \alpha_n - \beta_n - e_n)^2 \|x_n - p\|^2 + 2\alpha_n k \|x_{n+1} - p\|^2 \\ &\quad + 2\beta_n k \|x_{n+1} - p\|^2 + 2\alpha_n \delta_n M_1 + 2\beta_n \tau_n M_1 + 2e_n M, \end{aligned} \quad (2.7)$$

where  $\delta_n = \|T_1 y_n - T_1 x_{n+1}\| \rightarrow 0$  as  $n \rightarrow \infty$  and  $\tau_n = \|T_1 z_n - T_1 x_{n+1}\| \rightarrow 0$  as  $n \rightarrow \infty$ . But,

$$\begin{aligned}
 \|y_n - x_{n+1}\| &= \|(1 - a_n - b_n - e'_n)x_n + a_n T_2 z_n + b_n T_2 x_n + e'_n v_n \\
 &\quad - (1 - \alpha_n - \beta_n - e_n)x_n - \alpha_n T_1 y_n - \beta_n T_1 z_n - e_n u_n\| \\
 &= \|a_n(T_2 z_n - x_n) + b_n(T_2 x_n - x_n) + e'_n(v_n - x_n) \\
 &\quad + \alpha_n(x_n - T_1 y_n) + \beta_n(x_n - T_1 z_n) + e_n(x_n - u_n)\| \\
 &\leq a_n \|T_2 z_n - x_n\| + b_n \|T_2 x_n - x_n\| + e'_n \|v_n - x_n\| \\
 &\quad + \alpha_n \|x_n - T_1 y_n\| + \beta_n \|x_n - T_1 z_n\| + e_n \|x_n - u_n\| \\
 &\leq a_n M + b_n M + e'_n M + \alpha_n M_1 + \beta_n M_1 + e_n M_1 \\
 &= M(a_n + b_n + e'_n) + M_1(\alpha_n + \beta_n + e_n) \\
 &\leq M(a_n + b_n + e'_n + \alpha_n + \beta_n + e_n) \rightarrow 0,
 \end{aligned} \tag{2.8}$$

as  $n \rightarrow \infty$ .

$$\begin{aligned}
 \|z_n - x_{n+1}\| &= \|(1 - c_n - e''_n)x_n + c_n T_3 x_n + e''_n w_n - (1 - \alpha_n - \beta_n - e_n)x_n \\
 &\quad - \alpha_n T_1 y_n - \beta_n T_1 z_n - e_n u_n\| \\
 &= \|c_n(T_3 x_n - x_n) + e''_n(w_n - x_n) + \alpha_n(x_n - T_1 y_n) \\
 &\quad + \beta_n(x_n - T_1 z_n) + e_n(x_n - u_n)\| \\
 &\leq c_n \|T_3 x_n - x_n\| + e''_n \|w_n - x_n\| + \alpha_n \|x_n - T_1 y_n\| \\
 &\quad + \beta_n \|x_n - T_1 z_n\| + e_n \|x_n - u_n\| \\
 &\leq c_n M_2 + e''_n M + \alpha_n M_1 + \beta_n M_1 + e_n M \\
 &\leq M(c_n + e''_n + \alpha_n + \beta_n + e_n) \rightarrow 0,
 \end{aligned}$$

as  $n \rightarrow \infty$ . This implies that  $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|x_{n+1} - z_n\| = 0$  since  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $\lim_{n \rightarrow \infty} b_n = 0$ ,  $\lim_{n \rightarrow \infty} c_n = 0$ ,  $\lim_{n \rightarrow \infty} e'_n = 0$ ,  $\lim_{n \rightarrow \infty} e''_n = 0$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\lim_{n \rightarrow \infty} e_n = 0$ . Using the uniform continuity of  $T_1$ , we obtain  $\delta_n = \|T_1 y_n - T_1 x_{n+1}\| \rightarrow 0$  as  $n \rightarrow \infty$  and  $\tau_n = \|T_1 z_n - T_1 x_{n+1}\| \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, there exists a positive integer  $N$  such that  $\alpha_n, \beta_n < \min\{\frac{1}{2k}, \frac{1-k}{(1-k)^2+k^2}\}$  for all  $n \geq N$ . Hence, from (2.7), we obtain

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \left(\frac{(1-\alpha_n-\beta_n-e_n)^2}{1-2\alpha_n k-2\beta_n k}\right) \|x_n - p\|^2 + \frac{2\alpha_n \delta_n M_1 + 2\beta_n \tau_n M_1 + 2e_n M_1}{1-2\alpha_n k-2\beta_n k} \\
 &\leq \left(\frac{(1-\alpha_n)^2}{1-2\alpha_n k-2\beta_n k}\right) \|x_n - p\|^2 + M \left(\frac{2\alpha_n \delta_n + 2\beta_n \tau_n + 2e_n}{1-2\alpha_n k-2\beta_n k}\right) \\
 &= \left(\frac{1-2\alpha_n+\alpha_n^2-2\alpha_n k-2\beta_n k+2\alpha_n k+2\beta_n k}{1-2\alpha_n k-2\beta_n k}\right) \|x_n - p\|^2 \\
 &\quad + M \left(\frac{2\alpha_n \delta_n + 2\beta_n \tau_n + 2e_n}{1-2\alpha_n k-2\beta_n k}\right) \\
 &= \left(1 - \frac{2\alpha_n - \alpha_n^2 - 2\alpha_n k}{1-2\alpha_n k-2\beta_n k}\right) \|x_n - p\|^2 + M \left(\frac{2\alpha_n \delta_n + 2\beta_n \tau_n + 2e_n}{1-2\alpha_n k-2\beta_n k}\right) \\
 &= \left(1 - \frac{2-\alpha_n-2k}{1-2\alpha_n k-2\beta_n k} \alpha_n\right) \|x_n - p\|^2 + M \left(\frac{2\alpha_n \delta_n + 2\beta_n \tau_n + 2e_n}{1-2\alpha_n k-2\beta_n k}\right) \\
 &\leq (1 - (1-k)\alpha_n) \|x_n - p\|^2 + M \left(\frac{2\alpha_n \delta_n + 2\beta_n \tau_n + 2e_n}{1-2\alpha_n k-2\beta_n k}\right).
 \end{aligned} \tag{2.9}$$

Next, set  $\rho_n = \|x_n - p\|$ ,  $\lambda_n = (1-k)\alpha_n$  and  $\sigma_n = M \left(\frac{2\alpha_n \delta_n + 2\beta_n \tau_n + 2e_n}{1-2\alpha_n k-2\beta_n k}\right)$ . Using Lemma 1.2, we have  $\|x_n - p\| \rightarrow 0$  as  $n \rightarrow \infty$ . The proof of Theorem 2.1 is completed.  $\square$

**Corollary 2.2** Let  $E$  be a real Banach space,  $D$  a nonempty closed and convex subset of  $E$ . Let  $T_1, T_2, T_3$  be self maps of  $D$  with  $T_1(D)$  bounded such that  $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$  and  $T_1, T_2$  and  $T_3$  uniformly continuous. Suppose  $T_1, T_2, T_3$  are strongly pseudocontractive mappings. For  $x_0, u_0, v_0, w_0 \in D$ , the three step iteration with errors  $\{x_n\}$  defined as follows

$$\begin{cases} x_{n+1} = a_n x_n + b_n T_1 y_n + c_n u_n \\ y_n = a'_n x_n + b'_n T_2 z_n + c'_n v_n \\ z_n = a''_n x_n + b''_n T_3 x_n + c''_n w_n \end{cases} \quad n \geq 0, \tag{2.10}$$

where  $\{u_n\}, \{v_n\}$  and  $\{w_n\}$  are arbitrary bounded sequences in  $D$ .  $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{a''_n\}, \{b''_n\}$  and  $\{c''_n\}$  are real sequences in  $[0, 1]$  satisfying the following conditions:

- (i)  $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$
- (ii)  $b_n, b'_n, c_n, c'_n \rightarrow 0$  as  $n \rightarrow \infty$ .
- (iii)  $\sum_{n=1}^{\infty} b_n = \infty$
- (iv)  $\lim_{n \rightarrow \infty} \frac{c_n}{b_n} = 0$ ,

converges strongly to the unique common fixed point of  $T_1, T_2$  and  $T_3$ .

**Remark 2.3** Corollary 2.2 is Theorem 2.1 of Mogbademu and Olaleru [11]. Observe that if  $\beta_n = b_n \equiv 0$  for all  $n = 0, 1, 2, \dots$  in Theorem 2.1, then we obtain Theorem 2.1 of [11]. Similarly, if  $\beta_n = e_n = b_n = e'_n = e''_n \equiv 0$  for all  $n = 0, 1, 2, \dots$  in Theorem 2.1, then we obtain Theorem 2.1 of Xue and Fan [25]. Hence, Theorem 2.1 is an improvement and a generalization of Mogbademu and Olaleru [11], Xue and Fan [25] which in turn is a correction of Rafiq [20].

**Theorem 2.4** Let  $E$  be a real Banach space,  $T_1, T_2, T_3 : E \rightarrow E$  be uniformly continuous and strongly accretive operators with  $R(I - T_1)$  bounded, where  $I$  is the identity mapping on  $E$ . Let  $p$  denote the unique common solution to the equation  $T_i x = f$ , ( $i = 1, 2, 3$ ). For a given  $f \in E$ , define the operator  $H_i : E \rightarrow E$  by  $H_i x = f + x - T_i x$ , ( $i = 1, 2, 3$ ). For any  $x_0 \in E$ , the sequence  $\{x_n\}_{n=0}^{\infty}$  is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n - e_n)x_n + \alpha_n H_1 y_n + \beta_n H_1 z_n + e_n u_n, & n \geq 0, \\ y_n = (1 - a_n - b_n - e'_n)x_n + a_n H_2 z_n + b_n H_2 x_n + e'_n v_n, \\ z_n = (1 - c_n - e''_n)x_n + c_n H_3 x_n + e''_n w_n, \end{cases} \quad (2.11)$$

where  $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{e_n\}_{n=0}^{\infty}, \{e'_n\}_{n=0}^{\infty}, \{e''_n\}_{n=0}^{\infty}$  are real sequences in  $[0, 1]$  satisfying the conditions:  $a_n, b_n, e'_n, \alpha_n, \beta_n, e_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\alpha_n + \beta_n + e_n < 1$ ,  $a_n + b_n + e'_n < 1$ ,  $c_n + e''_n < 1$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$  are bounded sequences in  $E$ . Then the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to the unique common solution to  $T_i x = f$  ( $i = 1, 2, 3$ ).

**Proof.** Clearly, if  $p$  is the unique common solution to the equation  $T_i x = f$  ( $i = 1, 2, 3$ ), it follows that  $p$  is the unique common fixed point of  $H_1, H_2$  and  $H_3$ . Using the fact that  $T_1, T_2$  and  $T_3$  are all strongly accretive operators, then  $H_1, H_2$  and  $H_3$  are all strongly pseudocontractive with constant  $k = \max\{k_1, k_2, k_3\}$  where  $k_1, k_2, k_3 \in (0, 1)$  are strongly pseudocontractive constants for  $H_1, H_2$  and  $H_3$  respectively. Since  $T_i$  ( $i = 1, 2, 3$ ) is uniformly continuous with  $R(I - T_1)$  bounded, this implies that  $H_i$  ( $i = 1, 2, 3$ ) is uniformly continuous with  $R(H_1)$  bounded. Hence, Theorem 2.4 follows from Theorem 2.1.  $\square$

**Remark 2.5** Theorem 2.4 improves and extends Theorem 2.4 of Mogbademu and Olaleru [11] and Theorem 2.2 of Xue and Fan [25] which in turn is a correction of Rafiq [20].

**Example 2.6** Let  $E = (-\infty, +\infty)$  with the usual norm and let  $D = [0, +\infty)$ . We define  $T_1 : D \rightarrow D$  by  $T_1 x := \frac{x}{2(1+x)}$  for each  $x \in D$ . Hence,  $F(T_1) = \{0\}$ ,  $R(T_1) = [0, \frac{1}{2})$  and  $T_1$  is a uniformly continuous and strongly pseudocontractive mapping. Define  $T_2 : D \rightarrow D$  by  $T_2 x := \frac{x}{4}$  for all  $x \in D$ . Hence,  $F(T_2) = \{0\}$  and  $T_2$  is a uniformly continuous and strongly pseudocontractive mapping. Define  $T_3 : D \rightarrow D$  by  $T_3 x := \frac{\sin^4 x}{4}$  for each  $x \in D$ . Then  $F(T_3) = \{0\}$  and  $T_3$  is a uniformly continuous and strongly pseudocontractive mapping. Set  $\alpha_n = \frac{1}{n+1}$ ,  $\beta_n = \frac{1}{(n+1)+(n+1)^2}$ ,  $e_n = \frac{1}{(n+1)^2}$ ,  $a_n = \frac{1}{4(n+1)^{\frac{1}{2}}}$ ,  $b_n = \frac{1}{2(n+1)^2+(n+1)}$ ,  $e'_n = \frac{1}{(n+1)+(n+1)^2+(n+1)^3}$ ,  $c_n = \frac{1}{(n+1)^{\frac{1}{2}}}$ ,  $e''_n = \frac{1}{(n+1)^{\frac{1}{2}}+(n+1)^{\frac{1}{3}}}$ , for all  $n \geq 0$ . Clearly,  $F(T_1) \cap F(T_2) \cap F(T_3) = \{0\} = p \neq \emptyset$ . For an arbitrary  $x_0 \in D$ , the sequence  $\{x_n\}_{n=0}^{\infty} \subset D$  defined by (1.11) converges strongly to the common fixed point of  $T_1, T_2$  and  $T_3$  which is  $\{0\}$ , satisfying Theorem 2.1. This means that Theorem 2.1 is applicable.



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